The Point-Slope Form

7.3 OBJECTIVES

1. Given a point and a slope, find the graph of a line
2. Given a point and the slope, find the equation of a line
3. Given two points, find the equation of a line

Often in mathematics it is useful to be able to write the equation of a line, given its slope and any point on the line. In this section, we will derive a third special form for a line for this purpose.

Suppose that a line has slope \( m \) and that it passes through the known point \( P(x_1, y_1) \). Let \( Q(x, y) \) be any other point on the line. Once again we can use the definition of slope and write

\[
\frac{y - y_1}{x - x_1} = m
\]

Multiplying both sides of equation (1) by \( x - x_1 \), we have

\[
m(x - x_1) = y - y_1
\]

or

\[
y - y_1 = m(x - x_1)
\]

Equation (2) is called the point-slope form for the equation of a line, and all points lying on the line [including \((x_1, y_1)\)] will satisfy this equation. We can state the following general result.

Rules and Properties: Point-Slope Form for the Equation of a Line

The equation of a line with slope \( m \) that passes through point \((x_1, y_1)\) is given by

\[
y - y_1 = m(x - x_1)
\]

Example 1

Finding the Equation of a Line

Write the equation for the line that passes through point \((3, -1)\) with a slope of 3.

Letting \((x_1, y_1) = (3, -1)\) and \( m = 3 \) in point-slope form, we have

\[
y - (-1) = 3(x - 3)
\]

or

\[
y + 1 = 3x - 9
\]

We can write the final result in slope-intercept form as

\[
y = 3x - 10
\]

CHECK YOURSELF 1

Write the equation of the line that passes through point \((-2, -4)\) with a slope of \(\frac{3}{2}\). Write your result in slope-intercept form.
Because we know that two points determine a line, it is natural that we should be able to write the equation of a line passing through two given points. Using the point-slope form together with the slope formula will allow us to write such an equation.

**Example 2**

**Finding the Equation of a Line**

Write the equation of the line passing through (2, 4) and (4, 7).

First, we find \( m \), the slope of the line. Here

\[
m = \frac{7 - 4}{4 - 2} = \frac{3}{2}
\]

Now we apply the point-slope form with \( m = \frac{3}{2} \) and \((x_1, y_1) = (2, 4)\):

\[
y - 4 = \frac{3}{2}(x - 2)
\]

\[
y - 4 = \frac{3}{2}x - 3
\]

\[
y = \frac{3}{2}x + 1
\]

**NOTE** We could just as well have chosen to let \((x_1, y_1) = (4, 7)\). The resulting equation will be the same in either case. Take time to verify this for yourself.

**CHECK YOURSELF 2**

Write the equation of the line passing through \((-2, 5)\) and \((1, 3)\). Write your result in slope-intercept form.

A line with slope zero is a horizontal line. A line with an undefined slope is vertical. The next example illustrates the equations of such lines.

**Example 3**

**Finding the Equation of a Line**

(a) Find the equation of a line passing through \((7, -2)\) with a slope of zero.

We could find the equation by letting \( m = 0 \). Substituting the ordered pair \((7, -2)\) into the slope-intercept form, we can solve for \( b \).

\[
y = mx + b
\]

\[
-2 = 0(7) + b
\]

\[
-2 = b
\]

So,

\[
y = 0 \cdot x - 2 \quad \text{or} \quad y = -2
\]

It is far easier to remember that any line with a zero slope is a horizontal line and has the form

\[
y = b
\]
The value for $b$ will always be the $y$ coordinate for the given point.

(b) Find the equation of a line with undefined slope passing through $(4, -5)$.

A line with undefined slope is vertical. It will always be of the form $x = a$, in which $a$ is the $x$ coordinate for the given point. The equation is $x = 4$

**CHECK YOURSELF 3**

(a) Find the equation of a line with zero slope that passes through point $(-3, 5)$.

(b) Find the equation of a line passing through $(-3, -6)$ with undefined slope.

Alternate methods for finding the equation of a line through two points exist and have particular significance in other fields of mathematics, such as statistics. The following example shows such an alternate approach.

**Example 4**

**Finding the Equation of a Line**

Write the equation of the line through points $(-2, 3)$ and $(4, 5)$.

First, we find $m$, as before:

$$m = \frac{5 - 3}{4 - (-2)} = \frac{2}{6} = \frac{1}{3}$$

We now make use of the slope-intercept equation, but in a slightly different form. Because $y = mx + b$, we can write

$$b = y - mx$$

Now letting $x = -2$, $y = 3$, and $m = \frac{1}{3}$, we can calculate $b$:

$$b = 3 - \left(\frac{1}{3}\right)(-2)$$

$$= 3 + \frac{2}{3} = \frac{11}{3}$$

With $m = \frac{1}{3}$ and $b = \frac{11}{3}$, we can apply the slope-intercept form to write the equation of the desired line. We have

$$y = \frac{1}{3}x + \frac{11}{3}$$

**CHECK YOURSELF 4**

*Repeat the Check Yourself 2 exercise, using the technique illustrated in Example 4.*

We now know that we can write the equation of a line once we have been given appropriate geometric conditions, such as a point on the line and the slope of that line. In some applications the slope may be given not directly but through specified parallel or perpendicular lines.
Example 5

Finding the Equation of a Line

Find the equation of the line passing through \((-4, -3)\) and parallel to the line determined by \(3x + 4y = 12\).

First, we find the slope of the given parallel line, as before:

\[
3x + 4y = 12
\]

\[
4y = -3x + 12
\]

\[
y = -\frac{3}{4}x + 3
\]

Now because the slope of the desired line must also be \(-\frac{3}{4}\), we can use the point-slope form to write the required equation:

\[
y - (-3) = \frac{3}{4}[x - (-4)]
\]

This simplifies to

\[
y = -\frac{3}{4}x - 6
\]

and we have our equation in slope-intercept form.

**CHECK YOURSELF 5**

Find the equation of the line passing through \((5, 4)\) and perpendicular to the line with equation \(2x - 5y = 10\).

*Hint: Recall that the slopes of perpendicular lines are negative reciprocals of each other.*

The following chart summarizes the various forms of the equation of a line.

<table>
<thead>
<tr>
<th>Form</th>
<th>Equation for Line (L)</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>(ax + by = c)</td>
<td>Constants (a) and (b) cannot both be zero.</td>
</tr>
<tr>
<td>Slope-intercept</td>
<td>(y = mx + b)</td>
<td>Line (L) has (y) intercept ((0, b)) with slope (m).</td>
</tr>
<tr>
<td>Point-slope</td>
<td>(y - y_1 = m(x - x_1))</td>
<td>Line (L) passes through point ((x_1, y_1)) with slope (m).</td>
</tr>
<tr>
<td>Horizontal</td>
<td>(y = k)</td>
<td>Slope is zero.</td>
</tr>
<tr>
<td>Vertical</td>
<td>(x = h)</td>
<td>Slope is undefined.</td>
</tr>
</tbody>
</table>

**CHECK YOURSELF ANSWERS**

1. \(y = \frac{3}{2}x - 1\)
2. \(y = -\frac{2}{3}x + \frac{11}{3}\)
3. (a) \(y = 5\); (b) \(x = -3\)
4. \(y = -\frac{2}{3}x + \frac{11}{3}\)
5. \(y = -\frac{5}{2}x + \frac{33}{2}\)
7.3 Exercises

Write the equation of the line passing through each of the given points with the indicated slope. Give your results in slope-intercept form, where possible.

1. \((0, 2), m = 3\)
2. \((0, -4), m = -2\)
3. \((0, 2), m = \frac{3}{2}\)
4. \((0, -3), m = -2\)
5. \((0, 4), m = 0\)
6. \((0, 5), m = -\frac{3}{5}\)
7. \((0, -5), m = \frac{5}{4}\)
8. \((0, -4), m = -\frac{3}{4}\)
9. \((1, 2), m = 3\)
10. \((-1, 2), m = 3\)
11. \((-2, -3), m = -3\)
12. \((1, -4), m = -4\)
13. \((5, -3), m = \frac{2}{5}\)
14. \((4, 3), m = 0\)
15. \((2, -3), m \text{ is undefined}\)
16. \((2, -5), m = \frac{1}{4}\)

Write the equation of the line passing through each of the given pairs of points. Write your result in slope-intercept form, where possible.

17. \((2, 3) \text{ and } (5, 6)\)
18. \((3, -2) \text{ and } (6, 4)\)
19. \((-2, -3) \text{ and } (2, 0)\)
20. \((-1, 3) \text{ and } (4, -2)\)
21. \((-3, 2) \text{ and } (4, 2)\)
22. \((-5, 3) \text{ and } (4, 1)\)
23. (2, 0) and (0, −3)  
24. (2, −3) and (2, 4)

25. (0, 4) and (−2, −1)  
26. (−4, 1) and (3, 1)

Write the equation of the line $L$ satisfying the given geometric conditions.

27. $L$ has slope 4 and $y$ intercept (0, −2).

28. $L$ has slope $\frac{2}{3}$ and $y$ intercept (0, 4).

29. $L$ has $x$ intercept (4, 0) and $y$ intercept (0, 2).

30. $L$ has $x$ intercept (−2, 0) and slope $\frac{3}{4}$.

31. $L$ has $y$ intercept (0, 4) and a 0 slope.

32. $L$ has $x$ intercept (−2, 0) and an undefined slope.

33. $L$ passes through point (3, 2) with a slope of 5.

34. $L$ passes through point (−2, −4) with a slope of $\frac{3}{2}$.

35. $L$ has $y$ intercept (0, 3) and is parallel to the line with equation $y = 3x − 5$.

36. $L$ has $y$ intercept (0, −3) and is parallel to the line with equation $y = \frac{2}{3}x + 1$.

37. $L$ has $y$ intercept (0, 4) and is perpendicular to the line with equation $y = −2x + 1$.

38. $L$ has $y$ intercept (0, 2) and is parallel to the line with equation $y = −1$.

39. $L$ has $y$ intercept (0, 3) and is parallel to the line with equation $y = 2$.

40. $L$ has $y$ intercept (0, 2) and is perpendicular to the line with equation $2x − 3y = 6$.

41. $L$ passes through point (−3, 2) and is parallel to the line with equation $y = 2x − 3$.

42. $L$ passes through point (−4, 3) and is parallel to the line with equation $y = −2x + 1$.

43. $L$ passes through point (3, 2) and is parallel to the line with equation $y = \frac{4}{3}x + 4$. 
44. \( L \) passes through point \((-2, -1)\) and is perpendicular to the line with equation \( y = 3x + 1 \).

45. \( L \) passes through point \((5, -2)\) and is perpendicular to the line with equation \( y = -3x - 2 \).

46. \( L \) passes through point \((3, 4)\) and is perpendicular to the line with equation \( y = \frac{3}{5}x + 2 \).

47. \( L \) passes through \((-2, 1)\) and is parallel to the line with equation \( x + 2y = 4 \).

48. \( L \) passes through \((-3, 5)\) and is parallel to the \( x \) axis.

49. Describe the process for finding the equation of a line if you are given two points on the line.

50. How would you find the equation of a line if you were given the slope and the \( x \) intercept?

51. A temperature of 10\(^\circ\)C corresponds to a temperature of 50\(^\circ\)F. Also 40\(^\circ\)C corresponds to 104\(^\circ\)F. Find the linear equation relating \( F \) and \( C \).

52. In planning for a new item, a manufacturer assumes that the number of items produced \( x \) and the cost in dollars \( C \) of producing these items are related by a linear equation. Projections are that 100 items will cost $10,000 to produce and that 300 items will cost $22,000 to produce. Find the equation that relates \( C \) and \( x \).

53. A word processing station was purchased by a company for $10,000. After 4 years it is estimated that the value of the station will be $4000. If the value in dollars \( V \) and the time the station has been in use \( t \) are related by a linear equation, find the equation that relates \( V \) and \( t \).

54. Two years after an expansion, a company had sales of $42,000. Four years later the sales were $102,000. Assuming that the sales in dollars \( S \) and the time in years \( t \) are related by a linear equation, find the equation relating \( S \) and \( t \).

### Getting Ready for Section 7.4 [Section 2.7]

Graph each of the following inequalities.

- (a) \( x < 3 \)
- (b) \( x \geq -2 \)
- (c) \( 2x \leq 8 \)
- (d) \( 3x \geq -9 \)
- (e) \(-3x < 12 \)
- (f) \(-2x \leq 10 \)
- (g) \( \frac{2}{3}x \leq 4 \)
- (h) \( -\frac{3}{4}x \geq 6 \)
Answers

1. \( y = 3x + 2 \)  
3. \( y = \frac{3}{2}x + 2 \)  
5. \( y = 4 \)  
7. \( y = \frac{5}{4}x - 5 \)  
9. \( y = 3x - 1 \)  
11. \( y = -3x - 9 \)  
13. \( y = \frac{2}{5}x - 5 \)  
15. \( x = 2 \)  
17. \( y = x + 1 \)  
19. \( y = \frac{3}{4}x - \frac{3}{2} \)  
21. \( y = 2 \)  
23. \( y = \frac{3}{2}x - 3 \)  
25. \( y = \frac{5}{2}x + 4 \)  
27. \( y = 4x - 2 \)  
29. \( y = -\frac{1}{2}x + 2 \)  
31. \( y = 4 \)  
33. \( y = 5x - 13 \)  
35. \( y = 3x + 3 \)  
37. \( y = \frac{1}{2}x + 4 \)  
39. \( y = 3 \)  
41. \( y = 2x + 8 \)  
43. \( y = \frac{4}{3}x - 2 \)  
45. \( y = \frac{1}{3}x - \frac{11}{3} \)  
47. \( y = -\frac{1}{2}x \)  
49.  
51. \( F = \frac{9}{5}C + 32 \)  

53. \( V = -1500t + 10,000 \)

a.  
b.  
c.  
d.  
e.  
f.  
g.  
h.